Assignment 3. Solutions.

Problems. February 22.

1. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$.

   Solution.

   The vector we are looking for is $-3\frac{\mathbf{v}}{|\mathbf{v}|}$.

   We have
   $$|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

   Therefore,
   $$-3\frac{\mathbf{v}}{|\mathbf{v}|} = -2\sqrt{3}\mathbf{v} = -\sqrt{3}\mathbf{i} + \sqrt{3}\frac{1}{2}\mathbf{j} + \sqrt{3}\mathbf{k}.$$

2. Given $P_1(1, 4, 5)$ and $P_2(4, -2, 7)$, find

   a) the direction of $\overrightarrow{P_1P_2}$,

   b) the midpoint of the line segment P_1P_2.

   Solution.

   a) $\overrightarrow{P_1P_2} = \langle 3, -6, 2\rangle$
   $$|\overrightarrow{P_1P_2}| = \sqrt{9 + 36 + 4} = 7.$$

   Therefore, the direction of $\overrightarrow{P_1P_2}$ is
   $$\frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \langle \frac{3}{7}, \frac{-6}{7}, \frac{2}{7}\rangle.$$

   b) The midpoint of the line segment P_1P_2 is
   $$(\frac{1+4}{2}, \frac{4-2}{2}, \frac{5+7}{2}) = (\frac{5}{2}, 1, 6).$$

3. Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Write $\mathbf{u} = \mathbf{u_1} + \mathbf{u_2}$, where $\mathbf{u_1}$ is parallel to $\mathbf{v}$ and $\mathbf{u_2}$ is parallel to $\mathbf{w}$.

   Solution.

   Since $\mathbf{u_1}$ is parallel to $\mathbf{v}$ and $\mathbf{u_2}$ is parallel to $\mathbf{w}$, we have
   $$\mathbf{u_1} = a\mathbf{v} \qquad \text{and} \qquad \mathbf{u_2} = b\mathbf{w},$$

   for some numbers a and b.

   Thus,
   $$\mathbf{u} = a\mathbf{v} + b\mathbf{w},$$

$$\mathbf{u} = \mathbf{i} - 2\mathbf{j} = a(2\mathbf{i} + 3\mathbf{j}) + b(\mathbf{i} + \mathbf{j}) = (2a + b)\mathbf{i} + (3a + b)\mathbf{j}.$$

Equating the corresponding coefficients, we get a system of two equations with unknowns a and b.

$$\begin{cases} 2a + b = 1, \\ 3a + b = -2. \end{cases}$$

From here we get $a = -3$ and $b = 7$.

So, $u_1 = -6\mathbf{i} - 9\mathbf{j}$ and $u_2 = 7\mathbf{i} + 7\mathbf{j}$.

4. Find the coordinates of the point Q that divides the segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ into two lengths whose ratio is $p/q = r$.

   Solution.

   We are given $|P_1Q|/|QP_2| = r$. So, $|P_1Q| = r|QP_2|$. Adding $|QP_2|$ to both sides we get $|P_1P_2| = (r + 1)|QP_2|$.

   So,
$$|QP_2| = \frac{1}{r+1}|P_1P_2|.$$

   Since P_2Q is parallel to P_2P1, we see that

$$P_2Q = \frac{1}{r+1}P_2P_1 = \frac{1}{r+1}\langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle.$$

   So,

$$\overrightarrow{OQ} = \overrightarrow{OP_2} + \overrightarrow{P_2Q} = \langle x_2, y_2, z_2\rangle + \frac{1}{r+1}\langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle$$

$$= \langle\frac{1}{r+1}x_1 + \frac{r}{r+1}x_2, \frac{1}{r+1}y_1 + \frac{r}{r+1}y_2, \frac{1}{r+1}z_1 + \frac{r}{r+1}z_2\rangle.$$

   Thus, the coordinates of the point Q are

$$\left(\frac{x_1 + rx_2}{r+1}, \frac{y_1 + ry_2}{r+1}, \frac{z_1 + rz_2}{r+1}\right).$$

Problems. March 01.

1. $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$. Find

   a) $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$,

   b) the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$,

   c) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$,

   d) the vector projection $\text{proj}_{\mathbf{v}}\mathbf{u}$.

   Solution.

   a) $\mathbf{v} \cdot \mathbf{u} = -\sqrt{2} + \sqrt{3}$, $|\mathbf{v}| = \sqrt{2}$, $|\mathbf{u}| = 3$.

b) $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|\,|\mathbf{u}|} = \dfrac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$.

c) $\dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|} = \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$.

d) $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|^2}\mathbf{v} = \dfrac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$.

2. Find the measures of the angles between the diagonals of the rectangle whose vertices are $A(1,0)$, $B(0,3)$, $C(3,4)$, $D(4,1)$.

   Solution.

   The diagonals are $\overrightarrow{AC} = \langle 2,4\rangle$ and $\overrightarrow{BD} = \langle 4,-2\rangle$.

   $\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$. Therefore, the diagonals meet at $90°$.

3. $\mathbf{u} = \mathbf{j}+\mathbf{k}$, $\mathbf{v} = \mathbf{i}+\mathbf{j}$ Write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

   Solution.

$$\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u})$$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|^2}\mathbf{v} = \frac{1}{2}(\mathbf{i}+\mathbf{j}) = \frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}.$$

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{j}+\mathbf{k} - \left(\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right) = -\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}.$$

   So,

$$\mathbf{u} = \left(\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right) + \left(-\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}\right),$$

   where the first vector is parallel to $\mathbf{v}$ and the second vector is orthogonal to $\mathbf{v}$.

4. Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

   Solution.

   We have $\overrightarrow{CA} = -\mathbf{v}+(-\mathbf{u})$, $\overrightarrow{CB} = -\mathbf{v}+\mathbf{u}$. Then

$$\overrightarrow{CA}\cdot\overrightarrow{CB} = (-\mathbf{v}-\mathbf{u})\cdot(-\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2-|\mathbf{u}|^2 = 0,$$

   since both $\mathbf{u}$ and $\mathbf{v}$ are unit vectors.

   $\overrightarrow{CA}\cdot\overrightarrow{CB} = 0$ means that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

Problems. March 03.

1. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Find the length and direction of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

   Solution.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

   Length of this vector: $|\mathbf{u} \times \mathbf{v}| = \sqrt{4+4+4} = 2\sqrt{3}$.
   Its direction: $\dfrac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = -\dfrac{1}{\sqrt{3}}\,\mathbf{i} - \dfrac{1}{\sqrt{3}}\,\mathbf{j} + \dfrac{1}{\sqrt{3}}\,\mathbf{k}$.
   Since $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$, it has the same length $2\sqrt{3}$ and the opposite direction: $\dfrac{1}{\sqrt{3}}\,\mathbf{i} + \dfrac{1}{\sqrt{3}}\,\mathbf{j} - \dfrac{1}{\sqrt{3}}\,\mathbf{k}$.

2. $P(1,1,1)$, $Q(2,1,3)$, $R(3,-1,1)$.

   a) Find the area of the triangle determined by the points P, Q and R.

   b) Find a unit vector perpendicular to the plane PQR.

   Solution.
   $\overrightarrow{PQ} = \langle 1,0,2 \rangle$, $\overrightarrow{PR} = \langle 2,-2,0 \rangle$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = 6.$$

   The area of the triangle: $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = 3$.
   A unit vector perpendicular to the plane PQR:

$$\frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

   (or its opposite).

3. $\mathbf{u} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = -\mathbf{i} - \mathbf{k}$, $\mathbf{w} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and find the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

   Solution.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

   $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 8$.

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}.$$

$(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = 8.$

$$\mathbf{w}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -2 \\ 1 & 1 & -2 \end{vmatrix} = -6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

$(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v} = 8.$

The volume of the parallelepiped is 8.

4. Let $\mathbf{u} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{v} = -\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{w} = \mathbf{i}+\mathbf{k}$, $\mathbf{r} = -(\pi/2)\mathbf{i}-\pi\mathbf{j}+(\pi/2)\mathbf{k}$. Which vectors are

   a) perpendicular?

   b) parallel?

   Solution.

   a) $\mathbf{u}\cdot\mathbf{v} = 0$, $\mathbf{u}\cdot\mathbf{w} = 0$, $\mathbf{u}\cdot\mathbf{r} = -3\pi$, $\mathbf{v}\cdot\mathbf{w} = 0$, $\mathbf{v}\cdot\mathbf{r} = 0$, $\mathbf{w}\cdot\mathbf{r} = 0$. Vectors with zero dot product are mutually orthogonal.

   b) $\mathbf{u}$ and $\mathbf{r}$ are parallel since $\mathbf{r} = -\pi/2\mathbf{u}$. All other vectors cannot be parallel, since they are mutually orthogonal.

Problems. March 5.

1. Find parametric equations for the line through $P(1,2,-1)$ and $Q(-1,0,1)$.

   Solution.

   $\overrightarrow{PQ} = \langle -2,-2,2\rangle$. Therefore the line is given by the equations

$$\begin{aligned} x &= 1-2t, \\ y &= 2-2t, \qquad t\in\mathbb{R}. \\ z &= -1+2t, \end{aligned}$$

2. Find parametric equations for the line through the point $(3,2,-1)$ parallel to the line $x = 1+2t$, $y = 2-t$, $z = 3t$.

   Solution.

   The line given above is parallel to the vector $\langle 2,-1,3\rangle$. Therefore, the line we are looking for has equations

$$\begin{aligned} x &= 3+2s, \\ y &= 2-s, \qquad s\in\mathbb{R}. \\ z &= -1+3s, \end{aligned}$$

3. Find parametric equations for the line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$.

Solution.

The line is parallel to the plane's normal vector $\langle 3, 7, -5\rangle$. So, the line is given by the equations

$$\begin{aligned} x &= 2 + 3t, \\ y &= 4 + 7t, \qquad t \in \mathbb{R}. \\ z &= 5 - 5t, \end{aligned}$$

4. Find parametric equations for the line segment joining the points $(1, 0, -1)$ and $(0, 3, 0)$.

Solution.

$$\mathbf{r}(t) = t\langle 1, 0, -1\rangle + (1 - t)\langle 0, 3, 0\rangle = \langle t, 3 - 3t, -t\rangle,$$

where $0 \le t \le 1$.

5. Find the distance from the point $(2, 1, -1)$ to the line $x = 2t$, $y = 1+2t$, $z = 2t$.

Solution.

Denote $P(2, 1, -1)$ and $S(0, 1, 0)$. Clearly, S belongs to the line. Then $\overrightarrow{SP} = \langle 2, 0, -1\rangle$. Consider the vector $\mathbf{v} = \langle 2, 2, 2\rangle$, parallel to the line. The distance from P to the line equals

$$d = \frac{|\overrightarrow{SP} \times \mathbf{v}|}{|\mathbf{v}|}.$$

We have

$$\overrightarrow{SP} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}.$$

$$|\overrightarrow{SP} \times \mathbf{v}| = \sqrt{56} = 2\sqrt{14}.$$

$$|\mathbf{v}| = 2\sqrt{3}.$$

So,

$$d = \sqrt{\frac{14}{3}}.$$

Problems. March 8.

1. Find an equation of the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$.

   Solution.

   The desired plane has normal vector $\langle 3, 1, 1 \rangle$. Therefore, its equation is

$$3(x-1) + (y+1) + (z-3) = 0,$$

$$3x + y + z - 5 = 0.$$

2. Find the plane determined by the intersecting lines.

$$L_1 : x = t,\ y = 3 - 3t,\ z = -2 - t;\ t \in \mathbb{R},$$

$$L_2 : x = 1 + s,\ y = 4 + s,\ z = -1 + s;\ s \in \mathbb{R}.$$

   Solution.

   Let us find the point of intersection of these two lines. Equating the corresponding coordinates, we get the following three equations.

$$t = 1 + s, \quad 3 - 3t = 4 + s, \quad -2 - t = -1 + s.$$

   Adding 1st and 3rd equations we get $s = -1$, therefore $t = 0$. We also see that these two values satisfy 2nd equation. Putting $t = 0$ into the equations of line L_1, we get the point of intersection $(0, 3, -2)$.

   In order to obtain a vector $\mathbf{n}$ normal to the plane, we compute the cross product of the vectors $\langle 1, -3, -1 \rangle$ and $\langle 1, 1, 1 \rangle$ that are parallel to the given lines.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

   In fact, as a normal vector we can take another vector parallel to the latter one: $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

   Therefore, the plane has the equation

$$(x-0) + (y-3) - 2(z+2) = 0,$$

$$x + y - 2z - 7 = 0.$$

3. Find the distance from the point $(2, 2, 3)$ to the plane $2x + y + 2z = 4$.

   Solution.

$$d = \frac{|2 \cdot 2 + 1 \cdot 2 + 2 \cdot 3 - 4|}{\sqrt{4 + 1 + 4}} = \frac{8}{3}.$$

4. Find the angle between the planes $5x+y-z=10$ and $x-2y+3z=-1$.

Solution.

The angle between the planes is the acute angle between their normals $\mathbf{n}_1 = \langle 5, 1, -1\rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3\rangle$.

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_1|}{|\mathbf{n}_1|\ |\mathbf{n}_1|} = 0.$$

So, the planes are orthogonal (the angle is 90°).

Problems. March 10.

1. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k}$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at $t = \pi/6$. Write the particle's velocity at that time as the product of its speed and direction.

   Solution.

   Velocity: $\mathbf{v}(t) = (\sec t \tan t)\mathbf{i} + (\sec^2 t)\mathbf{j} + \frac{4}{3}\mathbf{k}$

   Acceleration: $\mathbf{a}(t) = (\sec t \tan^2 t + \sec^3 t)\mathbf{i} + (2\sec^2 t \tan t)\mathbf{j}$

   Velocity at $t = \pi/6$: $\mathbf{v}(\pi/6) = \frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$

   Speed at $t = \pi/6$: $|\mathbf{v}(\pi/6)| = \sqrt{\frac{4}{9} + \frac{16}{9} + \frac{16}{9}} = 2.$

   Direction at $t = \pi/6$: $\frac{\mathbf{v}(\pi/6)}{|\mathbf{v}(\pi/6)|} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

   Velocity at $t = \pi/6$ as the product of velocity and direction: $2\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

2. Evaluate the integral

$$\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t \cos t)\mathbf{k}]\, dt.$$

   Solution.

$$\begin{aligned}
&\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t \cos t)\mathbf{k}]\, dt \\
&= \Big[(\sec t)\mathbf{i} - (\ln\cos t)\mathbf{j} + (\sin^2 t)\mathbf{k}\Big]\Big|_0^{\pi/3} \\
&= (2-1)\mathbf{i} - (\ln\frac{1}{2} - \ln 1)\mathbf{j} + (\frac{3}{4} - 0)\mathbf{k} = \\
&= \mathbf{i} + \ln 2\mathbf{j} + \frac{3}{4}\mathbf{k}
\end{aligned}$$

3. Solve the initial value problem.

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$
$$\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k},$$
$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0}.$$

Solution. Integrating

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$

we get

$$\frac{d\mathbf{r}}{dt} = -(t\mathbf{i}+t\mathbf{j}+t\mathbf{k}) + \overrightarrow{C_1}.$$

Using

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0},$$

we see that $\overrightarrow{C_1} = \mathbf{0}$. So,

$$\frac{d\mathbf{r}}{dt} = -(t\mathbf{i}+t\mathbf{j}+t\mathbf{k}).$$

Integrating one more time, we get

$$\mathbf{r} = -\frac{1}{2}(t^2\mathbf{i}+t^2\mathbf{j}+t^2\mathbf{k}) + \overrightarrow{C_2}.$$

Using

$$\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k},$$

we see that

$$\overrightarrow{C_2} = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}.$$

Thus,

$$\mathbf{r} = (10 - \frac{t^2}{2})(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

4. Find parametric equations for the line that is tangent to the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$ at $t_0 = \pi/2$.

Solution.

$$\mathbf{r}(\pi/2) = \mathbf{j}$$

$$\mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2\cos 2t)\mathbf{k}$$
$$\mathbf{v}(\pi/2) = -\mathbf{i} - 2\mathbf{k}.$$

So, the tangent line is

$$\begin{aligned} x &= -s, \\ y &= 1, \qquad\qquad s \in \mathbb{R}. \\ z &= -2s, \end{aligned}$$

5. Show that the vector-valued function

$$\mathbf{r}(t) = (2\mathbf{i}+2\mathbf{j}+\mathbf{k})+\cos t\left(\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}\right)+\sin t\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right)$$

describes the motion of a particle moving in a circle of radius 1 centered at the point $(2, 2, 1)$ and lying in the plane $x + y - 2z = 2$.

Solution.

Parametric equations of the given curve are

$$\begin{aligned} x(t) &= 2 + \tfrac{1}{\sqrt{2}}\cos t + \tfrac{1}{\sqrt{3}}\sin t \\ y(t) &= 2 - \tfrac{1}{\sqrt{2}}\cos t + \tfrac{1}{\sqrt{3}}\sin t \\ z(t) &= 1 + \tfrac{1}{\sqrt{3}}\sin t \end{aligned}$$

Substituting these into the equation of the plane, we get

$$\left(2+\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{3}}\sin t\right)+\left(2-\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{3}}\sin t\right)-2\left(1+\frac{1}{\sqrt{3}}\sin t\right)=2.$$

This is true for all t. Thus the given curve lies in the given plane.

Now compute the distance from the point $(x(t), y(t), z(t))$ to the point $(2, 2, 1)$.

$$d = \sqrt{(x(t)-2)^2+(y(t)-2)^2+(z(t)-1)^2}$$

$$= \sqrt{\left(\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{3}}\sin t\right)^2+\left(-\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{3}}\sin t\right)^2+\left(\frac{1}{\sqrt{3}}\sin t\right)^2}$$

$$= \sqrt{\frac{1}{2}\cos^2 t+\frac{2}{\sqrt{6}}\cos t\sin t+\frac{1}{3}\sin^2 t+\frac{1}{2}\cos^2 t-\frac{2}{\sqrt{6}}\cos t\sin t+\frac{1}{3}\sin^2 t+\frac{1}{3}\sin^2 t}$$

$$= \sqrt{\cos^2 t+\sin^2 t} = 1.$$

So, all the points on the curve are at distance 1 from the given point. Thus, the curve is a circle of radius 1.

Problems. March 12.

1. Find the length of the curve

$$\mathbf{r}(t) = 2t\ \mathbf{i}+t^2\ \mathbf{j}+\frac{1}{3}t^3\ \mathbf{k}, \qquad 0 \le t \le 1.$$

Solution.

$$\mathbf{v}(t) = 2\ \mathbf{i}+2t\ \mathbf{j}+t^2\ \mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{4+4t^2+t^4} = 2+t^2.$$

$$L = \int_0^1 |\mathbf{v}(t)|dt = \int_0^1 (2+t^2)dt = (2t+\frac{1}{3}t^3)\Big|_0^1 = 2+\frac{1}{3} = \frac{7}{3}.$$

2. Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$\mathbf{r}(t) = e^{2t}\cos 2t\ \mathbf{i} + 2\ \mathbf{j} + e^{2t}\sin 2t\ \mathbf{k}.$$

Solution.

$$\mathbf{v}(t) = (2e^{2t}\cos 2t - 2e^{2t}\sin 2t)\mathbf{i} + (2e^{2t}\sin 2t + e^{2t}\cos 2t)\ \mathbf{k}.$$

$$|\mathbf{v}(t)| = \sqrt{(2e^{2t}\cos 2t - 2e^{2t}\sin 2t)^2 + (2e^{2t}\sin 2t + e^{2t}\cos 2t)^2}$$

$$= \sqrt{4e^{4t}\cos^2 2t - 8e^{4t}\cos 2t\sin 2t + 4e^{4t}\sin^2 2t + 4e^{4t}\sin^2 2t + 8e^{4t}\cos 2t\sin 2t + 4e^{4t}\cos^2 2t}$$

$$= \sqrt{8e^{4t}} = 2\sqrt{2}e^{2t}.$$

$$s = \int_0^t 2\sqrt{2}e^{2\tau}d\tau = \sqrt{2}e^{2\tau}\Big|_0^t = \sqrt{2}\left(e^{2t} - 1\right).$$

So,

$$e^{2t} = \frac{s}{\sqrt{2}} + 1,$$

$$2t = \ln(\frac{s}{\sqrt{2}} + 1),$$

$$t = \frac{1}{2}\ln(\frac{s}{\sqrt{2}} + 1).$$

Substituting this into $\mathbf{r}(t)$ and using that

$$e^{\ln(\frac{s}{\sqrt{2}}+1)} = \frac{s}{\sqrt{2}} + 1,$$

we get

$$\mathbf{r}(t(s)) = (\frac{s}{\sqrt{2}} + 1)\cos\ln(\frac{s}{\sqrt{2}} + 1)\ \mathbf{i} + 2\ \mathbf{j} + (\frac{s}{\sqrt{2}} + 1)\sin\ln(\frac{s}{\sqrt{2}} + 1)\ \mathbf{k}.$$

3. Find $\mathbf{T}$ and κ for the curve

$$\mathbf{r}(t) = t^2\ \mathbf{i} + (\sin t - t\cos t)\ \mathbf{j} + (\cos t + t\sin t)\ \mathbf{k}, \qquad t > 0.$$

Solution.

$$\mathbf{v}(t) = 2t\ \mathbf{i} + (t\sin t)\ \mathbf{j} + (t\cos t)\ \mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = t\sqrt{5}.$$

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{2}{\sqrt{5}}\ \mathbf{i} + \frac{\sin t}{\sqrt{5}}\ \mathbf{j} + \frac{\cos t}{\sqrt{5}}\ \mathbf{k}$$

$$\frac{d\mathbf{T}}{dt} = \frac{\cos t}{\sqrt{5}}\ \mathbf{j} - \frac{\sin t}{\sqrt{5}}\ \mathbf{k}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{\cos^2 t}{5} + \frac{\sin^2 t}{5}} = \frac{1}{\sqrt{5}}$$

$$\kappa = \left|\frac{d\mathbf{T}}{dt}\right| / |\mathbf{v}(t)| = \frac{1}{5t}.$$